SOLVING LINEAR FRACTIONAL PROGRAMMING PROBLEMS BY COMPUTER ALGORITHM

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ABSTRACT

In this paper, we study a new approach for solving linear fractional programming problem by converting it into a single Linear Programming Problem, which can be solved by using any type of linear fractional programming technique. In the objective function of a linear fractional programming problem, if β is negative, the available methods are failed to solve, while our proposed method is capable of solving such problems. In the present paper, we propose a new method and develop 'FORTRAN' programs to solve the problem. The optimal linear fractional programming problem solution procedure is illustrated with numerical examples and also by a computer program. We also compare our method with other available methods for solving linear fractional programming problems.

Keywords: *Linear Programming, Linear Fractional Programming Problem, Computer Program.*

INTRODUCTION

Various optimization problems in engineering and economics involve maximization (or minimization) of the ratio of physical or economic function, for instances cost/time, cost/volume, cost/benefit, profit/cost or other quantities measuring the efficiency of the system. Naturally, there is a need for generalizing the simplex technique for linear programming to the ratio of linear functions or to the case of the ratio of quadratic functions in such a situation. All these problems are fragments of a general class of optimization problems, termed in the literature as fractional programming problems. This field of Linear Fractional Programming was developed by Hungarian mathematician Matros in 1960. Several methods are proposed to solve this problem. Charnes and Kooper have proposed their method depended on transforming this, Linear Fractional Programming Problem to an equivalent linear program, they say the feasible region X is nonempty and bounded, cx +α and ax+β do not vanish simultaneously in S then they used the variable transformation y= tx, t ≥ 0 in such a way that dt + β= γ where γ ≠ 0 is a specified number and transform linear fractional programming to a linear programming problem. Multiplying the numerator and denominator and the system of inequalities by t and y= tx, t ≥ 0, they obtain two equivalent Linear Programming problems and name them as EP and EN. If EP or EN has an optimal solution and other is inconsistence, then linear fractional programming also has an optimal solution. If any of the two problems EP or EN is unbound, then linear fractional programming is also unbound. On the other hand, the simplex type algorithm is introduced by Swarup and Swarup, Gupta and Mohan.

In that method one needs to compute $\Delta_{j=} Z^2 (c_{j - z_j}{}^{1})-Z^1 (d_j-Z_j{}^{2})$
In each step and continues this process until the value of j satisfying the required condition. We see that it has to deal with the ratio of two linear functions, that's why its computational process is complicated and also when the constraints are not in canonical form then it becomes lengthy. But to solve a sequence of problems sometimes may need many iterations and at some cases say, $dx + \beta \geq 0$ and $cx + \alpha < 0\ \forall x \in S$, Bitran-Novae method is failed. Singh in his paper makes a useful study about the optimality condition in fractional programming. Tantawy develops a technique with the dual solution.

In this paper, our intent is to develop a new technique for solving any type of Linear Fractional Programming problem by converting it into a single linear programming problem because at some cases in the denominator and numerator when β is negative, available methods are failed to solve the linear fractional problem. We also develop a FORTRAN computer program for solving it and analyse the solution by numerical examples.

MATHEMATICAL FORMULATION OF LINEAR PROGRAMMING AND LINEAR FRACTIONAL PROGRAMMING

The mathematical expression of a general linear programming problem is as follows:

$$\text{Maximize (or Minimize)} \quad Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \{\leq, =, \geq\} b_i\,;\ \ i = 1, 2, \cdots, m$$

$$x \geq 0; b \geq 0$$

Where one and only one of the signs (≤, =, ≥) holds for each constraint and the sign may vary from one constraint to another. Here c_j (j = 1, 2,…n) are called profit (or cost) coefficients, x_j (j= 1, 2,…n) are called decision variables. The set of feasible solution to the linear programming problem is

$S=\{(x_1,x_2\ldots,x_n)^T :(x_1,x_2,\ldots.x_n) \varepsilon\ R^n\}$ and $(x_1,x_2,\ldots,x_n)^T$

The set S is called the constraints set, feasible set or feasible region of Linear Programming.
In matrix vector notation the above problem can be expressed as:
Maximize (or Minimize) $Z=cx$
Subject to Ax (≤, =, ≥) b
$X \geq 0;\ b \geq 0$
where A is an $m\times n$ matrix, x is an $n\times 1$ column vector, b is an $m\times 1$ column vector and c is a $1\times n$ row vector, $b\in R^m$; $x,c\in R^n$.

The mathematical formulation of a Linear Fractional Programming (in matrix vector notation) is as follows:

$$\text{Maximize} \quad Z = \frac{cx+\alpha}{dx+\beta}$$

$$\text{Subject to} \quad Ax(\leq, =, \geq)b$$

$$x \geq 0; b \geq 0$$

where A is an $m \times n$ matrix, x is an $n \times 1$ column vector, b is an $m \times 1$ column vector and c is a $1 \times n$ row vector, $b \in R^m$; $x,c,d \in R^n$; $\alpha, \beta \in R$.

Solving Linear Fractional Programming by Our Method

If $c \neq 0$ and $d \neq 0$, we assume that the feasible reason $\{x \in R^n: Ax \leq b, x \geq 0\}$ is nonempty and bounded and the denominator $dx + \beta \neq 0$.

We convert the Linear Fractional Programming into a Linear Programming in the following way assuming that $\beta \neq 0$.

Case 1: $\beta > 0$

For objective function,

$$Z = \frac{cx+\alpha}{dx+\beta} = \frac{cx+\alpha}{dx+\beta} - \frac{\alpha}{\beta} + \frac{\alpha}{\beta} = \frac{(c\beta - d\alpha)x}{\beta(dx+\beta)} + \frac{\alpha}{\beta}$$

$$\therefore H(y) = Iy + J$$

where, $I = \dfrac{c\beta - d\alpha}{\beta}$, $y = \dfrac{x}{dx+\beta}$, $J = \dfrac{\alpha}{\beta}$

For constraint, $Ax \leq b$

$$\Rightarrow \frac{Ay\beta}{1-yd} \leq b$$

$$\Rightarrow y \leq \frac{b}{A\beta + bd}$$

$$\Rightarrow (A\beta + bd)y \leq b$$

$$\therefore Ky \leq L$$

where $K = A\beta + bd, L = b$

So the new LP is: Maximize $H(y) = Iy + J$

Subject to $Ky \leq L$, $y \geq 0$

CALCULATION FOR THE UNKNOWN VARIABLE OF THE LINEAR FRACTIONAL PROGRAMMING

Form the above Linear Programming, we get

$$y = \frac{x}{dx+\beta} \qquad \therefore x = \frac{y\beta}{1-dy}$$

This is our required optimal solution. Putting the value of x in the original objective function, we can get the optimal value.

Case 2: $\beta < 0$, $\alpha \geq 0$

For objective function,

$$Z = \frac{cx+\alpha}{dx-\beta}$$

$$\Rightarrow \frac{Z+1}{Z-1} = \frac{cx+\alpha+dx-\beta}{cx+\alpha-dx+\beta} = \frac{(c+d)x+\alpha-\beta}{(c-d)x+\alpha+\beta} = \frac{c'x+\alpha'}{d'x+\beta'}$$

where, $\alpha' = \alpha - \beta$, $\beta' = \alpha + \beta$ and $c' = c + d$, $d' = c - d$

Same as above procedure, we have

$$H'(y) = I'y + J'$$

where, $I' = \frac{c'\beta' - d'\alpha'}{\beta'}$, $y = \frac{x}{d'x+\beta'}$, $J' = \frac{\alpha'}{\beta'}$, $H'(y) = \frac{Z+1}{Z-1}$

For constraints, following the same procedure as above, we get

$$K'y \leq L'$$

where $K' = A\beta' + bd'$, $L' = b$

Case 3: β < 0, α < 0

For objective function:

$$Z=\frac{cx-\alpha}{dx-\beta}=\frac{-cx+\alpha}{-dx+\beta}$$

Same as above procedure, we have

$$H_1(y)=I_1y+J_1$$

where, $I_1=\dfrac{d\alpha-c\beta}{\beta}$, $y=\dfrac{x}{-dx+\beta}$, $J_1=\dfrac{\alpha}{\beta}$

For constraints, following the same procedure as above, we get

$$K_1y\leq L_1$$

Where $K_1= A\beta\text{-}bd$, $L_1=b$.

ALGORITHM

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If $\beta > 0$ then
        $I\leftarrow\frac{c\beta-d\alpha}{\beta}; y\leftarrow\frac{x}{dx+\beta}; J\leftarrow\frac{\alpha}{\beta}; K\leftarrow A\beta+bd; L\leftarrow b$
    $H(y)\leftarrow Iy+J$;  for all  $Ky\leq L\ \&\ y\geq 0$;  else if  $\beta<0\ \&\ \alpha\geq 0$  then
            $\alpha'\leftarrow\alpha-\beta; \beta'\leftarrow\alpha+\beta; c'\leftarrow c+d; d'\leftarrow c-d;$
            $I'\leftarrow\frac{c'\beta'-d'\alpha'}{\beta'}; y\leftarrow\frac{x}{d'x+\beta'}; J'\leftarrow\frac{\alpha'}{\beta'};$
            $K'\leftarrow A\beta'+bd'\ ; L'\leftarrow b$
    $H'(y)\leftarrow I'y+J'$;  for all  $K'y\leq L'\ \&\ y\geq 0$;
            $H(y)\leftarrow\frac{H'(y)+1}{H'(y)-1}$
else
        $I_1\leftarrow\frac{d\alpha-c\beta}{\beta}; y\leftarrow\frac{x}{-dx+\beta}; J_1\leftarrow\frac{\alpha}{\beta}; K_1\leftarrow A\beta-bd\ ; L_1\leftarrow b$
    $H(y)\leftarrow I_1y+J_1$, for all  $K_1y\leq L_1\ \&\ y\geq 0$;  return  $H(y)$.
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NUMERICAL EXAMPLES

Here we illustrate some numerical examples to demonstrate our method

Example: 1

$$\text{Minimize } Z=\frac{-2x_1+x_2+2}{x_1+3x_2+4}$$

$$\text{Subject to } -x_1+x_2\le 4$$

$$2x_1+x_2\le 14$$

$$x_2\le 6$$

$$x_1,x_2\ge 0$$

Solution: Here we have, $c=(-2,1), d=(1,3), \alpha=2, \beta=4, A_1=(-1,1), A_2(2,1), A_3=(0,1), b_1=4, b_2=14$, $b_3=6$, where A_1,b_1 is related to the first constraint, A_2,b_2 is related to the second constraint and A_3,b_3 is related to the third constraint. So, we have the new objective function.

$$\text{Minimize } H(y)=\left[\frac{4(-2,1)-2(1,3)}{4}\right]\begin{bmatrix}y_1\\y_2\end{bmatrix}+\frac{2}{4}=-\frac{5}{2}y_1-\frac{1}{2}y_2+\frac{1}{2}$$

Now for the first constraint,

$$\left[(-1,1)4+4(1,3)\right]\begin{bmatrix}y_1\\y_2\end{bmatrix}\le 4$$

$$\Rightarrow y_2\le\frac{1}{4}$$

For the second constraint,

$$\left[4(2,1)+14(1,3)\right]\begin{bmatrix}y_1\\y_2\end{bmatrix}\le 14$$

$$\Rightarrow 11y_1+23y_2\le 7$$

For the third constraint,

$$\left[4(0,1)+6(1,3)\right]\begin{bmatrix}y_1\\y_2\end{bmatrix}\le 6$$

$$\Rightarrow 3y_1+11y_2\le 3$$

Converting the LP in standard form we have

$$\text{Maximize } T(y)=-\min H(y)=\frac{5}{2}y_1+\frac{1}{2}y_2-\frac{1}{2}$$

$$\text{Subject to } y_2+s_1=\frac{1}{4}$$

$$11y_1+23y_2+s_2=7$$

$$3y_1+11y_2+s_3=3$$

$$y_1,y_2,s_1,s_2\ge 0$$

Now we get the following Table (Table A and Table B):

Table A. initial table for Example 1

C_B	C_j	$\frac{5}{2}$	$\frac{1}{2}$	0	0	0	b
	Basis	Y_1	Y_1	S_1	S_2	S_3	
0	S_1	0	1	1	0	0	$\frac{1}{4}$
0	S_2	11	23	0	0	0	7
0	S_3	3	11	0	0	1	3
$\bar{c}_j=c_j-E_j$		$\frac{5}{2}$	$\frac{1}{2}$	0	0	0	

Table B. Final table for Example:

C_B	c_j	$\frac{5}{2}$	$\frac{1}{2}$	0	0	0	b
	Basis	Y_1	Y_1	S_1	S_2	S_3	
0	S_1	0	1	1	0	0	$\frac{1}{4}$
$\frac{5}{2}$	Y_1	1	$\frac{23}{11}$	0	$\frac{1}{11}$	0	$\frac{7}{11}$
0	S_3	0	$\frac{52}{11}$	0	$-\frac{3}{11}$	1	$\frac{12}{11}$
$\bar{c}_j=c_j$-E_j		0	$-\frac{52}{11}$	0	$-\frac{5}{22}$	0	

So we have, $y_1 =\frac{7}{11}$, y_2=0

$$\text{Now } (x_1,x_2)=\frac{(y_1,y_2)\beta}{1-(1,3)(y_1,y_2)}=\frac{\left(\frac{7}{11},0\right)4}{1-(1,3)\left(\frac{7}{11},0\right)}=\frac{\left(\frac{28}{11},0\right)}{\frac{11}{4}}=(7,0)$$

Putting this value in the original objective function, we have

$$\text{Min } \mathbb{Z}=\frac{-2\times7+0+2}{7+3\times0+4} = -\frac{12}{11}$$

Now, we solve the above problem by computer program.

Output:

```
Minimum value of the Objective Function = −1.090909
    X1 =7.000000;
    X2  =0.000000.
```

We see that our hand calculation result and computer oriented solution is the same. This shows that our computer program is correct

	Charnes and Kooper	**Bitran and Novae**	**Swarup**	**Our Method**
EN	1 iteration and inconsistent	1st LP iterations	2 iterations with clumsy calculations of LFP	2 iterations with easy calculations of LP
EP	2 iterations in phase 1 1 iterations in Phase 2	2nd LP-2 iterations		

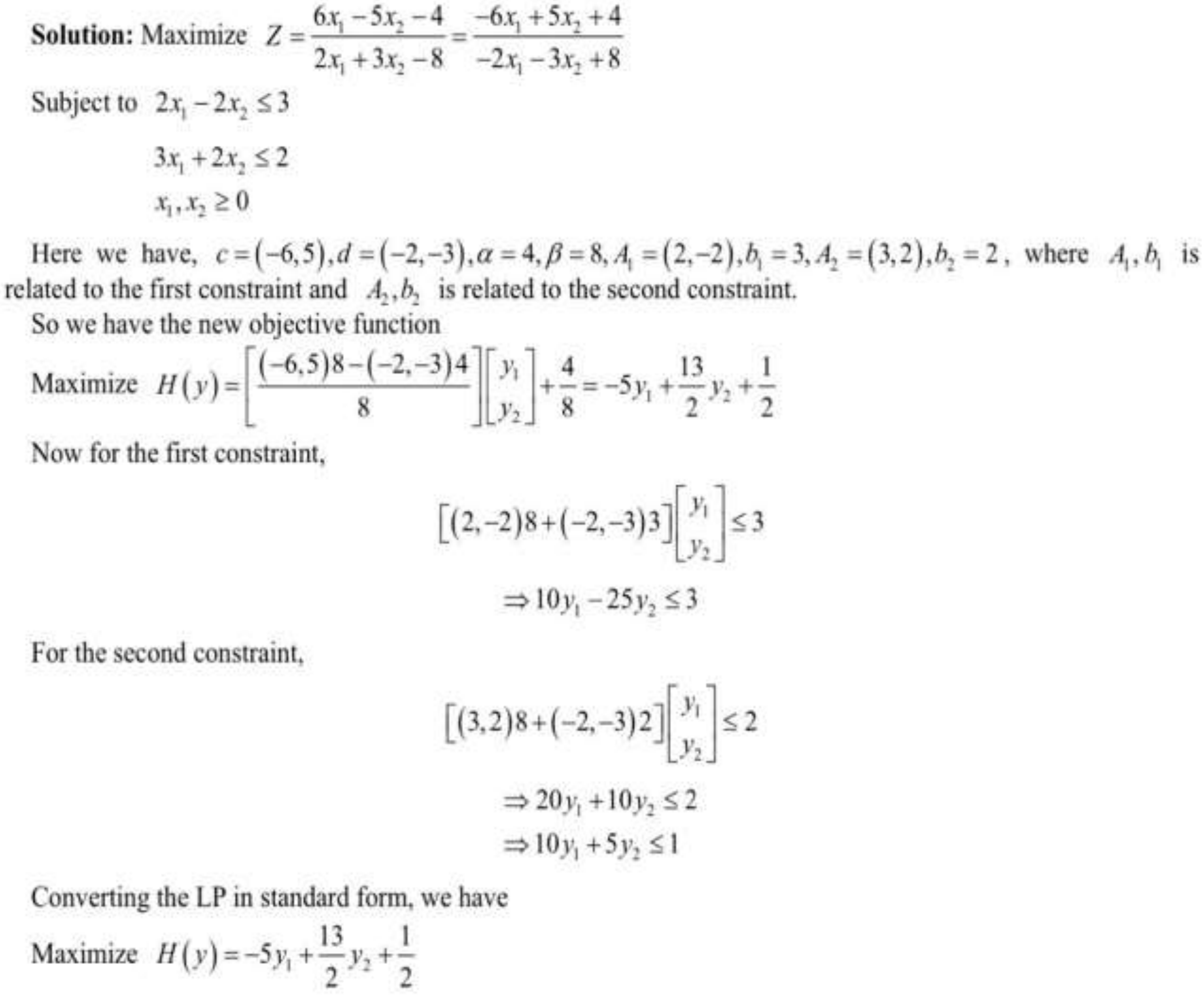
Example 2:

$$\text{Maximize} \quad Z = \frac{6x_1 - 5x_2 - 4}{2x_1 + 3x_2 - 8}$$

Subject to $2x_1$-$2x_2 \leq 3$

$3x_1$+$2x_2 \leq 2$

$x_1, x_2 \geq 0$



Solution: Maximize $Z = \frac{6x_1 - 5x_2 - 4}{2x_1 + 3x_2 - 8} = \frac{-6x_1 + 5x_2 + 4}{-2x_1 - 3x_2 + 8}$

Subject to $2x_1 - 2x_2 \leq 3$

$3x_1 + 2x_2 \leq 2$

$x_1, x_2 \geq 0$

Here we have, $c = (-6,5), d = (-2,-3), \alpha = 4, \beta = 8, A_1 = (2,-2), b_1 = 3, A_2 = (3,2), b_2 = 2$, where A_1, b_1 is related to the first constraint and A_2, b_2 is related to the second constraint.

So we have the new objective function

Maximize $H(y) = \left[\frac{(-6,5)8 - (-2,-3)4}{8}\right]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{4}{8} = -5y_1 + \frac{13}{2}y_2 + \frac{1}{2}$

Now for the first constraint,

$$\left[(2,-2)8 + (-2,-3)3\right]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq 3$$

$$\Rightarrow 10y_1 - 25y_2 \leq 3$$

For the second constraint,

$$\left[(3,2)8 + (-2,-3)2\right]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq 2$$

$$\Rightarrow 20y_1 + 10y_2 \leq 2$$

$$\Rightarrow 10y_1 + 5y_2 \leq 1$$

Converting the LP in standard form, we have

Maximize $H(y) = -5y_1 + \frac{13}{2}y_2 + \frac{1}{2}$

Subject to $10y_1$ -$25y_2$ + s_1 =3

$10y_1$ + $5y_1$ + S_2 =1

$y_1, y_2, s_1, s_2 \geq 0$

Now we get the following tables (Table **C** and Table **D**):

Table C. initial table for Example 2.

C_n	C_j	**-5**	$\frac{13}{2}$	**0**	**0**	***b***
	Basis	y_1	y_2	s_1	s_2	
0	s_1	10	-25	1	0	3
0	y_2	10	5	0	1	1
$\overline{c}_j = c_j - E_j$		-5	$\frac{13}{2}$	0	0	

Table D. Final table for Example 2.

C_B	C_j	**-5**	$\frac{13}{2}$	**0**	**0**	***b***
	Basis	y_1	y_2	s_1	s_2	
0	s_1	40	0	1	5	8
$\frac{13}{2}$	y_1	2	1	0	$\frac{1}{5}$	$\frac{1}{5}$
$\overline{c}_j = c_j - E_j$		-18	0	0	$\frac{-13}{10}$	

So we have $y_1 = 0$, $y_2 = \frac{1}{5}$

$$\text{Now, } (x_1, x_2) = \frac{(y_1, y_2)\beta}{1-(-2,-3)(y_1, y_2)} = \frac{\left(0, \frac{1}{5}\right)8}{1-(-2,-3)\left(0, \frac{1}{5}\right)} = \frac{\left(0, \frac{8}{5}\right)}{\frac{8}{5}} = (0,1)$$

Putting this value in the original objective function, we have

$$\text{Maximum} \quad Z = \frac{6\times0-5\times1-4}{2\times0+3\times1-8} = \frac{9}{5}$$

Using computer program, we get the following result.

Output:

```
Maximum value of the Objective Function = 1.800000.
X1 = 0.000000;
X2 = 0.000000.
```

COMPARISON

In this section, we compare our method with all other available methods and we find that our method is better than any other available method. The reasons are as follows:

* We can solve any type of linear fractional programming problems by this methodology.

* We can easily transfer the LFP problem into a LP problem.

* Its computational steps are so easy from other methods.

* In this method, we need to solve one LP but by other methods one needs to solve more than one LP, and thus our method saves valuable time.

* The final result converges quickly in this method.

* In this method there is one restriction that is $\beta \neq 0$.

*In some cases of the denominator and numerator say, $dx + \beta > 0$ and $cx + \alpha < 0 \; \forall \; x \in X$, where Btran - Novaes method fails and for the negative value of β all other existing methods are also failed, but our method is able to solve the problem very easily.

* Using computer program, we get the optimal solution of the linear fractional programming problem very quickly.

CONCLUSION

In this paper, we have provided a new method for solving linear fractional programming problem. While all other existing methods are failed in the case of negative value of β, but our method can solve the problem vary easily. At first we transform the linear fractional programming problems into a linear programming and then solve it by using simplex method. Then we develop a computer program for solving such problems, and verify that our computer program is correct. We illustrate a number of numerical examples to demonstrate our method. After that we compare our method with other existing methods. We further conclude that the proposed concept will be helpful in solving real-life problems involving linear fractional programming problems in agriculture, production planning, financial and corporate planning, health care, hospital management, etc. Thus our newly developed method with computer program saves time and energy and is easy to apply.

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